Generalization of Fibonacci Sequence and Related Polynomials

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[1] INTRODUCTION :

Leonardo Fibonacci was a mathematical innovator of the thirteenth century. He was born in Pisa, Italy around 1175 and was also known as Leonardo Pisano or Leonardo of Pisa. Fibonacci made many contributions to mathematics but is best known by laypersons for the sequence of numbers (0, 1, 1, 2, 3, 5, 8, 13, 18, 21, 34 …) which carries his name Fibonacci & associated with the breeding of rabbits.

The interesting Fibonacci numbers come up surprisingly often in nature and art as well as in science and engineering. They arise in computer algorithms, especially in sorting or organizing data. However, they also occur in Pascal's triangle, in Pythagorean triples, some areas of algebra, graph theory, quasicrystals and many other areas of mathematics.

The Fibonacci sequence can be found everywhere in nature, therefore, it can be found in man.

The Fibonacci numbers arise from the famous "rabbits problem" which is now nearly 750 years old and which to this day remains one of the most fascinating chapters of elementary mathematics.

The famous work done by Italian mathematician Leonardo of Pisa of Liber abaci (A book about abacus) and also better known by the Fibonacci (Fibonacci – a contraction of filius Bonacci, that is, son of Boncci). This book written in the year 1202, has survived in its second version, which appeared in 1228.

Liber abaci is a comprehensive work containing almost all of the arithmetic and algebraic knowledge of the day, and it played an important role in the development of Mathematics.

How many pairs of rabbits are created by one pair in one year?

"A certain man had one pair of rabbits together in a certain enclosed place and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also."

SOLUTION :

1. At the end of month 0, there is only one pair , and they are not yet breeding.
2. At the end of month 1, the first pair gives birth to a second pair.
3. At the end of month 2, the second pair is one month old and not yet breeding. The first pair produces another pair. Therefore, there are three pairs at the end of month second.

4. At the end of month 3, the first pair has produces yet another pair, and the second pair has given birth to its first progeny. Therefore, there are five pairs at the end of month three.

This leads to the sequence of numbers which the eminent French number theorist, Edouard Lucas, named in the 1870's the Fibonacci sequence. The Fibonacci is as in order

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 …

Fibonacci numbers in Nature

Fibonacci numbers are also appear in nature, from the leaf arrangements in the plants to the pattern of a flower.

- 3 Petals : Lilies
- 5 Petals : Buttercups, Roses
- 8 Petals : Delphinium
- 13 Petals : Marigolds
- 21 Petals : Black-eyed susans
- 55/89 Petals : Daisies

The Fibonacci sequence even shows up in Music, there are 8 white keys in an Octave and 5 black keys for a total of 13.

The Fibonacci numbers are studied as a part of number theory and have application in the counting of mathematical objects such as sets, permutations, sequence and to computer science.

THE FIBONACCI SEQUENCE

The Fibonacci sequence is a series of numbers in which the next number in the sequence is determined by adding the previous two.

Fibonacci sequence is defined by the recursion formula

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \qquad (1)$$

with $F_0 = 0, F_1 = 1$,where F_n is the n^{th} Fibonacci number.

The linear recursion relation (1) has a characteristic equation

$$x^2 - x - 1 = 0 \qquad (2)$$

The roots of equation (2) are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

Using these roots, we have

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (3)$$

It is called the Binet formula for Fibonacci numbers. It was derived by *Jacques Phillippe Marie Binet* (1786-1856), although the result was known to *Euler, Daniel Bernoulli and De Moivre.*

Closed form expression for Fibonacci number's Generating function is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \quad (4)$$

Francois Edouard Anatole Lucas (1842-1891) was the 19^{th} century French mathematician defined the same recurrence equation with different initial terms in May of 1876 as

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \quad (5)$$

with $L_0 = 2, L_1 = 1$,where L_n is the n^{th} Lucas number.

The Binet formula for the Lucas number is defined by

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} = \alpha^n + \beta^n. \quad (6)$$

Closed form expression for Fibonacci number's Generating function is

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2} \quad (7)$$

THE FIBONACCI-LIKE SEQUENCES

Bicknell **[3]** defined the Pell sequence P_n, by the second order linear recurrence sequence as $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ (8)

with $P_0 = 0, P_1 = 1$,where P_n is the n^{th} Pell number.

Cerin and Gianella **[6]** defined the Pell-Lucas sequence P_n, by the second order linear recurrence sequence as $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ (9)

with $P_0 = 2, P_1 = 2$,where P_n is the n^{th} Pell-Lucas number.

Koken and Bozkurt **[15]** defined Jacobsthal and Jacobsthal-Lucas sequence, by the second

order linear recurrence sequence.

Jacobsthal Sequence J_n , is define as $J_n = J_{n-1} + 2J_{n-2}$, $n \geq 2$ (10)

with $J_0 = 0, J_1 = 1$,where J_n is the n^{th} Jacobsthal number.

Jacobsthal-Lucas sequence j_n, is define as $j_n = j_{n-1} + 2j_{n-2}$, $n \geq 2$ (11)

with $j_0 = 2, j_1 = 1$,where j_n is the n^{th} Jacobsthal-Lucas number.

Feinberg **[8]** defined the Tribonacci sequence T_n , by the third order linear recurrence sequence as $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, $n \geq 0$ (12)

with $T_0 = 0, T_1 = T_2 = 1$,where T_n is the n^{th} Tribonacci number.

Jaiswal **[13]** defined the sequence Q_n , by the fourth order linear recurrence sequence as

$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3}$, $n \geq 4$ (13)

where Q_0, Q_1, Q_2, Q_3 are given arbitrary integers.

Similarly by changing the recurrence relation and different initial condition we get other sequences, which are known as Fibonacci-like sequence.

GENERALIZED FIBONACCI SEQUENCE

The Fibonacci sequence have been generalized by so many authors as following.

1. By changing the initial terms and maintaining the recurrence relation.
2. By vary the recurrence relation and maintaining the initial terms.
3. By modifying the recurrence relation so that each term is the sum of three or more preceding terms.

Many mathematicians have studied Fibonacci and generalized Fibonacci sequences for many years. The Fibonacci sequence has been generalized in several ways.

The few generalization of Fibonacci sequence is as follows:

Coleman **[7]** has shown interesting generalization of the Fibonacci sequence by using the general rule $T_{n+2} = T_n + BT_{n+1}$ (14)

where B is positive integer and T_n is the n^{th} term of the sequence.

Horadam **[11, 12]** defined the generalized Fibonacci sequence $\{W_n\}$ as

$W_n = pW_{n-1} - qW_{n-2}$, n≥2 (15)

where $W_0 = a, W_1 = b$ and a, b, p, q, are integers.

Waddill **[26]** has extended the Fibonacci recurrence relation and defined the sequence

$\{K_n\}$; $\quad K_n = K_{n-1} + K_{n-2} + K_{n-3}, n \geq 3$ (16)

where K_0, K_1, K_2 are arbitrary integers.

FIBONACCI POLYNOMIALS

The Fibonacci polynomials $\{f_n(x)\}$ are defined by

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x) \text{ with } f_1(x) = 1, f_2(x) = x. \quad (17)$$

It is easy to verify that the relation

$$f_{-n}(x) = (-1)^{n+1} f_n(x)$$

extends the definition of Fibonacci polynomials to all negative subscripts.

The generating function for Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} f_n(x)t^n = \frac{t}{1 - xt - t^2} \quad (18)$$

The Lucas polynomials $\{l_n(x)\}$ are defined by

$$l_{n+1}(x) = xl_n(x) + l_{n-1}(x) \text{ with } l_0(x) = 2, l_1(x) = x. \quad (19)$$

Lucas polynomials can be extended to negative subscripts by

$$l_{-n}(x) = (-1)^n l_n(x).$$

The generating function for Lucas polynomials is given by

$$\sum_{n=0}^{\infty} l_n(x)t^n = \frac{2 - xt}{1 - xt - t^2} \quad (20)$$

Bivariate Fibonacci polynomials are defined as

$$F_n(x, y) = xF_{n-1}(x, y) + yF_{n-2}(x, y), \quad F_0(x, y) = 0, F_1(x, y) = 1. \quad (21)$$

and bivariate Lucas polynomials as

$$L_n(x, y) = xL_{n-1}(x, y) + yL_{n-2}(x, y), \quad L_0(x, y) = 2, L_1(x, y) = x \quad (22)$$

GENERALIZED FIBONACCI POLYNOMIALS

The Fibonacci polynomials is generalized by

$$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x), n \geq 0 \text{ and } g_0(x) = 0, g_1(x) = 1, \quad (23)$$

where a and b are arbitrary integers.

Catalani [5] define the generalized bivariate Fiboncci polynomial as

$$H_n(x,y) = xH_{n-1}(x,y) + yH_{n-2}(x,y),\ H_0(x,y) = a_0, H_1(x,y) = a_1. \quad (24)$$

If we set $a_0 = 0, a_1 = 1$ we obtain the bivariate Fibonacci polynomials $F_n(x,y)$; with $a_0 = 2, a_1 = x$ we get the bivariate Lucas polynomials $L_n(x,y)$. The univariate polynomials are obtained by setting y = 1.

Following research work is proposed to carry out

1. Some generalization and extension of the Fibonacci sequence, Fibonacci-like sequence, Fibonacci polynomials and Fibonacci-Like polynomials will be made.
2. Some properties of generalized Fibonacci sequence and Fibonacci polynomials have been discussed and new identities will be established.
3. We will arrange standard results and new results of Fibonacci sequence, Fibonacci-Like sequence and there polynomials in Hypergeometric representations.
4. We will establish Connection Formulae, Mixed Formulae for Fibonacci sequence and Fibonacci polynomials with other sequences and polynomials.
5. Generalized Fibonacci and Fibonacci-Like polynomials introduced and derive some recurrence relations.
6. Some Trigonometric property of Fibonacci sequence, other classical sequence and their polynomials will be made.
7. We will extend Fibonacci polynomials in two variables and more than two variables. Also defined properties same as Fibonacci polynomials of one variables.

8. Vector identities will be obtained for generalized Fibonacci and Fibonacci-Like sequence.
9. We will establish some applications of Fibonacci numbers in Artificial Intelligence.

[2] A BRIEF REVIEW OF THE WORK ALREADY DONE IN THE FIELD

Macfarlane, A.J. **[16]** use the property for determinants and give new identities involving Fibonacci and related numbers. Koken and Bozkurt **[15]** define Jacobsthal M-matrix and Jacobsthal Q-matrix similar to Fibonacci Q-matrix and using these matrix representations to found the Binet like formula for Jacobsthal numbers.

Spivey **[24]** use the sum property for determinants and give a new proof of the famous identities like Cassini identity, d'Ocagne identity, and Catalan identity.

Fibonacci and generalized Fibonacci identities have been studied by many mathematicians for many years.

Horadam **[11]** suggests that there are two main directions in which the sequence may be generalized, either the recurrence relation can be preserved, but the first two Fibonacci numbers are replaced by arbitrary integers. He further suggested that these two techniques can be combined.

He defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2}, n \geq 3 \quad \text{with} \quad H_1 = p \text{ and } H_2 = p + q \qquad (25)$$

where p and q are arbitrary integers.

Waddill **[26]** has adopted the later technique. He extended the recurrence relation and has replaced the initial terms by arbitrary integers.

He defined the sequence $\{K_n\}$ by

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, n \geq 3 \qquad (26)$$

where K_0, K_1, K_2 are arbitrary integers.

Kalman and Mena **[14]** generalize the Fibonacci and Lucas sequence by

$$F_n = aF_{n-1} + bF_{n-2},\ n \geq 2 \quad with \quad F_0 = 0,\ F_1 = 1 \tag{27}$$

$$L_n = aL_{n-1} + bL_{n-2},\ n \geq 2 \quad with \quad L_0 = 2,\ L_1 = 1 \tag{28}$$

[3] NOTEWORTHY CONTRIBUTION IN THE FIELD OF PROPOSED WORK

Fibonacci numbers have been used with remarkable success with talented mathematics student from elementary school through graduate level mathematics. The Fibonacci and Lucas numbers appear in numerous mathematical problems. Vorobiev **[25]**, Alfred **[4]**, and Hoggatt **[10]** have given many interesting properties of the Fibonacci and the Lucas numbers.

Thousands of properties, results and theorems related to Fibonacci sequence have been discovered. In fact there is an entire journal dedicated to this subject *The Fibonacci Quarterly* is mathematical journal devoted entirely to the subject of mathematics related to Fibonacci sequence.

The *Fibonacci Quarterly* founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980) and Br. Alfred Brousseau (1907-1988). It is an official journal of the Fibonacci Association devoted to the study of integers with special properties.

B.A. Read **[21]** has used the Fibonacci series to predict the distance of the moons of Jupiter, Saturn & Uranus from their respective primary. He has shown that a particular moon's position is dependent upon the positions of the previous two moons closer to the primary. The Fibonacci numbers have been found in many other areas as well. In the area of Physics, there are indications that the Golden Ratio and Fibonacci numbers are related to the structure of atoms and also to the spacing of planets in the Solar system.

[4] PROPOSED METHODOLOGY DURING THE TENURE OF THE RESEARCH WORK

We will ahead our research in following way:

1. *Collection of material*: We will collect the material related to research topic by considering the reference books, doctorial thesis, articles, Journals

and Internet. The subject material will also be collected from various libraries.

2. *Discussion and Selection of Materials*: we will discuss with distinguished Professors and concerning the facts and compile all information collected from different sources.
3. *Organization*: We will organize our work in two ways, first we select the usable information and then we try to modify our result in right way.
4. *Framing an outline*: The Fibonacci sequence is obtained by adding two previous terms with two specific initial values. Similarly Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence, Jacobsthal-Lucas sequence etc., are also obtained by adding previous terms. We can generalize these sequences either change the initial values or change the recurrence relation. We shall establish and derive some of its identities and properties with the help of various techniques like Mathematical Induction, Determinant and Matrix properties.
5. *Presentation*: Finally we will set up the collected information, facts and ideas in thesis through an appropriate manner.

[5] EXPECTED OUTCOME OF THE PROPOSED RESEARCH WORK

As an outcome of our proposed research work, we have study generalization of Fibonacci sequence and related polynomials. Firstly, we will introduce generalized Fibonacci sequence. We will obtain fundamental properties and identities for these sequences. We will also obtain matrix and determinant identities. Moreover, we obtain vector identities for these sequences. We will establish generalized Fibonacci and Fibonacci-Like polynomials and derive recurrence relations. We will obtain matrix and determinant identities for generalized Fibonacci and Fibonacci-Like polynomials. We will establish relations between related polynomials. Moreover, we will establish results related to multivariate polynomials.

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